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The day when risk theory prevented a *Civil War*

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ABSTRACT The present piece *is not* a research paper on actuarial sciences, nor intends to be one. It represents the written form of a workshop that the author gives to his students from a first course on Risk Theory when teaching the topic of compound-type random variables under the assumptions of the model of collective risks. All the statistical hypotheses made here were specifically conceived for this presentation and should be carefully analyzed in any real-world application.

Key words
Model of collective risks, Wright model, Marvel Cinematic Universe, Detective Comics Extended Universe.

1. INTRODUCTION/WARNING

The purpose of this paper is to present an exercise on the model of collective risks as introduced in Chapter 17 from the book by Promislow (2015) (see also Chapter 12 in the book by Bowers *et al.* (1997)). Concretely, we assume the usual hypotheses on the model and aim for the Poisson-compound-Gamma random variable of aggregate claims, as referred to in the books by Klugman *et al.* (2019) and Stoltzfus and Dalton (2010).

This document is addressed to an actuarial audience that enjoys superhero films and comic books. Therefore, we strongly recommend the reader to watch *Man of Steel*, by Roven *et al.* (2013); *The Avengers*, by Feige (2012); and *Captain America: Civil War*, by Feige (2016) before reading this paper. Moreover, we caution the reader that we will extensively utilize the resolutions of the movies mentioned above. For this reason, consider this paragraph a *spoiler alert*.

The remainder of the paper is divided as follows. The next section presents a hypothetical argument between

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two superheroes about the monetary costs of their actions. Then, in section 3, we discuss the typical rationale given for the disagreements that cause the superheroes to clash with one another, and we also devise an (expensive) alternative for their conflict. Section 4 provides yet another (actuarial) possibility, along with its numeric implementation. Finally, section 5 gives our conclusions and poses a question regarding a typical extension of the material presented here. Appendices A and B give details on the Gamma and Negative Binomial distributions, respectively. If the reader intends to see the assumptions of the model of collective risks, its implementation on a spreadsheet program, and the computation of its expectation and distribution function, we would recommend they go straight to sections 4 and 5.

2. A *MULTIVERSAL* DEBATE

"Time. Space. Reality. It is more than a linear path. It's a prism of endless possibility. Where a single choice can branch out into infinite realities, creating alternate worlds from the ones you know. Each is a reflection of what could have been. Some heroes will rise, others will fall. And nothing will be the same. I am the Watcher. I am your guide through these vast new realities. Follow me and dare to face the unknown, and ponder the question... What if?". Uatu, the Watcher in the *What if?* series by Bradley (2021). See figure 1.

Figure 1 The Watcher, as depicted by (Byrne 2005, p.46).

What if there was a crossover universe such that, in the aftermath of the events shown in *Man of Steel* and *The Avengers*, Superman asked Tony Stark if it was necessary to have six Avengers fighting the *Chitauri* army in the third act of the movie? (After all, Kal-El defeated several Kryptonians single-handedly in his film.) Tony, being an expert debater, would then argue that they needed only two Avengers, while the remaining four would limit the destruction of the city to a few blocks[2]. See figure 2.

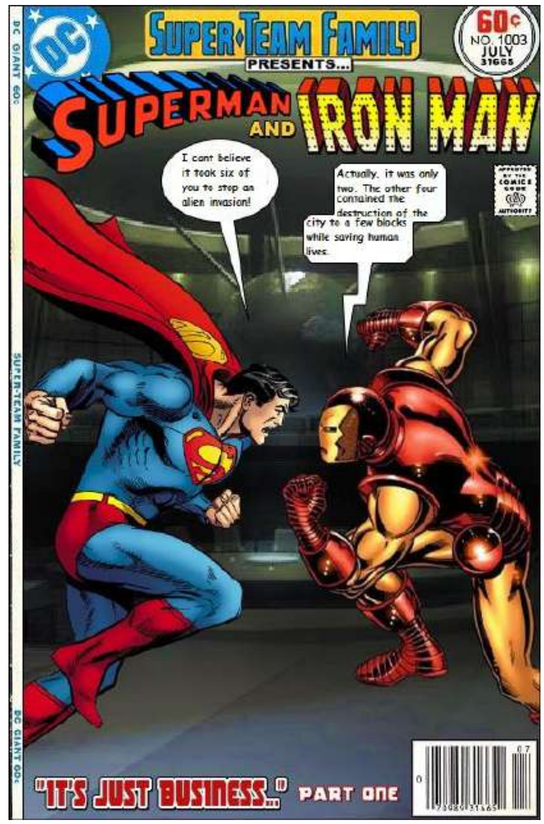

Figure 2 Steel meets Iron in an interesting debate. Modified version of the fanart by Pearsall (2015).

After saying this, Mr. Stark would inquire the Last Son of Krypton about the devastation he caused to the city of Metropolis in *Man of Steel*. Indeed, according to Zakarin (2013), about 129,000 people lost their lives, 250,000 people went missing, and a million people got injured during Superman's *heroic deeds*. In the aforementioned article, the author explicitly compares the impact on Metropolis to that of *Fat Man* on Nagasaki, Japan during the bombing

[2] A careful analysis of the sequence under scrutiny yields that there were always two Avengers fighting the alien army on top of the skyscrapers, but they alternated this task with those of containing the wrecking of the city and saving human lives.

of August 9, 1945. Monetarily, Clark Kent's *superheroic actions* exceed USD750 billion only in physical damage[3], and the overall cost of the catastrophe endured by the city of Metropolis exceeds two trillion USD.

3. SOKOVIA ACCORDS II

According to Rosza *et al.* (2016), *Captain America: Civil War* explores the internal conflict that arises among the Avengers when the United Nations proposes a system of accountability to address the potential collateral damage caused by their actions. This initiative leads to the signing of the so-called *Sokovia Accords* by the world's governments. The resulting *status quo* creates a deep rift within the team: Captain America argues that superheroes must remain free to act without governmental control, while Iron Man firmly supports regulation and oversight. In the comic book version by Millar *et al.* (2007), the storyline centers on the U.S. government's enactment of a *Superhero Registration Act*, which requires super-powered individuals to operate under official regulation, much like members of law enforcement. See figure 3.

Figure 3 The disagreement among the heroes due to a political action leads to a confrontation. Source material original from the works of Feige (2016) (above), and Millar *et al.* (2007) (below).

In both its representations (film and comic book), Marvel's Civil War is, by no means, the only example of an event where an institutional prohibition spawns a conflict between paladins of justice. The works by Moore *et al.* (1986); Miller *et al.* (1986); Walker (2004); Gordon *et al.* (2009) and Roven and Snyder (2016) are other well-known cases of the same situation. See figures 4-5.

Figure 4 The *Watchmen* series is so influential, it continues to affect pop culture even nowadays. These images are original from the works by Moore *et al.* (1986) (above) and Gordon *et al.* (2009) (below).

Figure 5 The film *Batman v Superman: Dawn of Justice* is heavily inspired (among others) by the comic book series *Batman: The Dark Knight Returns*. These images are original from the works by Miller *et al.* (1986) (left) and Roven and Snyder (2016) (right).

[3] Other comparisons offered by Zakarin (2013) are the estimated costs of the physical damages caused by The Avengers to Manhattan, and by the events of 9/11 to the United States of America. That is, USD160 billion, and USD55 billion, respectively.

Say the governments of the world agree to have the superheroes registered using what we could dub as *the Sokovia accords II*. In view of the previous discussion, we could expect to have a face-off between factions of heroes.

An (expensive) alternative

The governments of the world would, of course, look to prevent losing human lives. But they would also try to save money! A possibility to avoid the confrontation among our champions could arise from the following crossover conversation between Batman (BM) and Iron Man (IM). (See figure 6.)

BM– Hey, Tony! Would you be willing to cover the expenses for the first USD350 billion per event?

IM– Sure, Bruce! But only if you agree to pay for the casualties from USD350 billion to one trillion dollars. What do you say?

BM– Sounds good. Do you think the governments of the world would agree to cover the costs in excess of one trillion?

Figure 6 Batman and Iron Man talking about things. Image obtained from 9gag.com.

The answer to the final question displayed in the last paragraph depends, of course, on the probability that the governments of the world end up putting money of their own. However, a drawback of the proposed solution is that it would turn the (exciting) battles displayed in figures 2, 3 and 5 into the (not-so-exciting) battle displayed in figure 7.

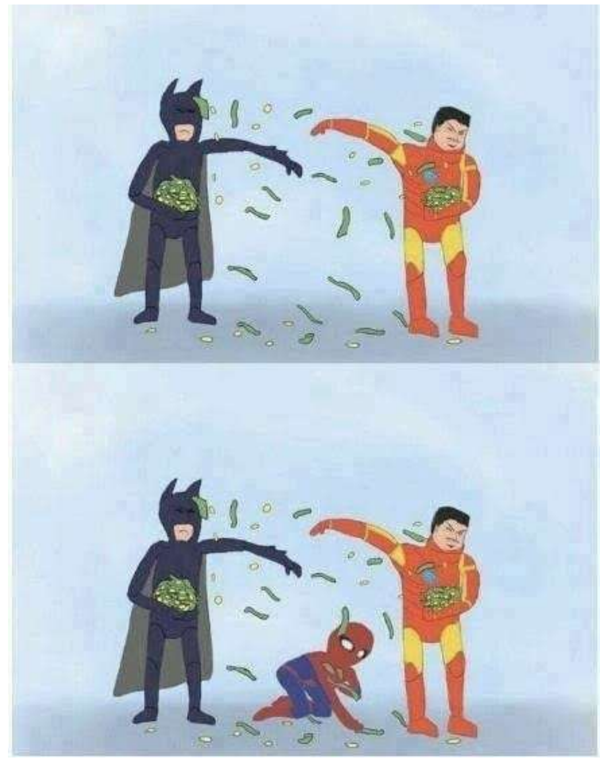

Figure 7 Not even the ~~wealthiest~~ noblest heroes should go against their own wealth. Image obtained from humorgeeky.com

Imagine that an actuary overhears the talk between BM and IM. You will surely agree that our heroes could turn to an insurance company to take on their respective risks. The following section delves into the details of the computation of the expected costs for all three of our protagonists[4]: Tony Stark, Bruce Wayne, and the governments of the world, and their respective probabilities of ending up paying more than those numbers.

4. THE *WRIGHT* ALTERNATIVE

Let N be a random variable with support on the set $\{0, 1, \ldots\}$. We will use N to describe *how many* casualties are linked to the superheroic actions of our clients, and call it a *frequency random variable*. On the other hand, define X_i as the non-negative random variable that mea-

[4] A more appropriate term would be *agents*, and in the case that they signed insurance contracts, they would become *policyholders*.

sures *how much* the i-th claim costs, for $i = 1, 2, ...$ Moreover, we assume the following hypotheses (see Chapter 17 in the book by Promislow (2015), and Chapter 12 in the book by Bowers *et al.* (1997)).

H1. The sequence of severity random variables $(X_i : i = 1, 2, ...)$ is independent of the frequency random variable N.

H2. The random variables follow a common distribution law, say F_X. We say that X is a *severity random variable*.

H3. The random variables X_i and X_j are pairwise independent for $i, j = 1, 2, ...$ and $i \neq j$.

Define now the random variable of aggregate claims S as

$$S := \begin{cases} 0 & \text{if } N = 0, \\ X_1 + \cdots + X_N & \text{if } N > 0. \end{cases} \tag{1}$$

The theorem of total probability

In the context of our assumptions, it is not hard to prove that

$$\mathbb{E}S = \mathbb{E}X \cdot \mathbb{E}N. \tag{2}$$

Indeed, using the theorem of nested expectation (see, for instance, formula (2.2.10) in the book by Bowers *et al.* (1997)), we see that:

$$\begin{aligned}
\mathbb{E}S &= \mathbb{E}[\mathbb{E}(S|N)] \\
&= 0 \cdot \mathbb{P}(N=0) + \sum_{n=1}^{\infty} \mathbb{E}\left[\sum_{k=1}^{N} X_k | N = n\right] \mathbb{P}(N=n) \\
&= 0 + \sum_{n=0}^{\infty} \mathbb{E}\left[\sum_{k=0}^{n} X_k\right] \mathbb{P}(N=n) \\
&= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \mathbb{E}X_k\right] \mathbb{P}(N=n) \\
&= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \mathbb{E}X\right] \mathbb{P}(N=n) \\
&= \sum_{n=0}^{\infty} n\mathbb{E}X\mathbb{P}(N=n) \\
&= \mathbb{E}X \sum_{n=0}^{\infty} n\mathbb{P}(N=n) = \mathbb{E}X \cdot \mathbb{E}N.
\end{aligned}$$

The second equality is just an application of (1) and the theorem of total probability; the third equality above holds because of the independence between N and the sequence of severity random variables $(X_i : i = 1, 2, ...)$ referred to in **H1**. The fifth equality is true because the severity random variables follow the common distribution of the random variable X (recall assumption **H2**).

As for the variance of the random variable of aggregate claims, note that

$$\mathbf{var}S = \mathbb{E}S^2 - (\mathbb{E}S)^2. \tag{3}$$

Now, an application of the theorem of total probability and (1) yield

$$\begin{aligned}
\mathbb{E}S^2 &= \sum_{n=0}^{\infty} \mathbb{E}\left[S^2 | N = n\right] \mathbb{P}(N=n) \\
&= 0 \cdot \mathbb{P}(N=0) + \sum_{n=1}^{\infty} \mathbb{E}\left[S^2 | N = n\right] \mathbb{P}(N=n).
\end{aligned}$$

Now, since

$$\mathbb{E}\left[S^2 | N = n\right] = \mathbf{var}(S|N=n) + [\mathbb{E}(S|N=n)]^2,$$

we can write

$$\begin{aligned}
\mathbb{E}S^2 &= \sum_{n=1}^{\infty} \left(\mathbf{var}(S|N=n) + [\mathbb{E}(S|N=n)]^2\right) \mathbb{P}(N=n) \\
&= \sum_{n=1}^{\infty} \left(\mathbf{var}\left(\sum_{k=1}^{n} X_k\right) + \left[\mathbb{E}\left(\sum_{k=1}^{n} X_k\right)\right]^2\right) \mathbb{P}(N=n) \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \mathbf{var}X_k + \left[\sum_{k=1}^{n} \mathbb{E}X_k\right]^2\right) \mathbb{P}(N=n) \\
&= \sum_{n=1}^{\infty} \left(n\mathbf{var}X + (n\mathbb{E}X)^2\right) \mathbb{P}(N=n) \\
&= \mathbf{var}X \sum_{n=1}^{\infty} n\mathbb{P}(N=n) + (\mathbb{E}X)^2 \sum_{n=1}^{\infty} n^2 \mathbb{E}N \\
&= \mathbf{var}X \cdot \mathbb{E}N + (\mathbb{E}X)^2 \mathbb{E}N^2.
\end{aligned} \tag{4}$$

The second equality above follows from **H1**; the third, from **H3**; and the fourth, from **H2**. Substituting (2) and (4) into (3) gives us

$$\begin{aligned}
\mathbf{var}S &= \mathbf{var}X \cdot \mathbb{E}N + (\mathbb{E}X)^2 \cdot \mathbb{E}N^2 - (\mathbb{E}X \cdot \mathbb{E}N)^2 \\
&= \mathbf{var}X \cdot \mathbb{E}N + (\mathbb{E}X)^2 \left(\mathbb{E}N^2 - (\mathbb{E}N)^2\right) \\
&= \mathbf{var}X \cdot \mathbb{E}N + (\mathbb{E}X)^2 \mathbf{var}N.
\end{aligned} \tag{5}$$

We can use the theorem of total probability again to find the cumulative distribution function of the random variable of aggregate claims:

$$\begin{aligned}
\mathbb{P}(S \leq s) &= \sum_{n=0}^{\infty} \mathbb{P}(S \leq s | N = n) \mathbb{P}(N=n) \\
&= \sum_{n=0}^{\infty} \mathbb{P}\left[\sum_{k=0}^{n} X_k \leq s\right] \mathbb{P}(N=n) \qquad (6) \\
&= \sum_{n=0}^{\infty} F_X^{*n}(s) \mathbb{P}(N=n), \qquad (7)
\end{aligned}$$

where, if $n > 0$, we let $F_X^{*n}(\cdot)$ denote the cumulative distribution function of the n-th convolution of the independent and identically random variables $X_1 + X_2 + \cdots + X_n$; otherwise, $F_X^{*n}(\cdot) = 0$. As was the case with (2), (6) holds by **H1**. On the other hand, (7) is true because of **H2** and **H3**.

A very popular choice for modeling the random variable of aggregate claims under assumptions **H1**-**H3** is the model introduced by Wright (1990), where the frequency random variable follows a Poisson distribution, and the severity is a Gamma-like random variable[5]. (For more on the Gamma distribution, see Appendix A and appendix A.3 in the book by Klugman *et al.* (2019).)

Spread*cheat*ing

For illustrative pruposes, but in line with the figures given by Zakarin (2013), let us assume that the Poisson parameter for the frequency distribution of the group of heroes is $\lambda = 7$, while the shape and scale parameters for their severity distribution are $\alpha = 18$ and $\theta = 6$. Since $\mathbb{E}N = \lambda = \mathbf{var}N$, and by (2) and (16), this results in an expected loss of

$$\mathbb{E}S = 18 \times 6 \times 7 = 756. \tag{8}$$

Turning to (5) and (17), also see that

$$\mathbf{var}S = (18 \times 6^2) \cdot 7 + (18 \times 6)^2 \cdot 7 = 86,184. \tag{9}$$

We will use the conversation between IM and BM to split the expected loss displayed in (8) into the three agents mentioned before: Iron Man, Batman, and the government of the world. To do this, we open our favorite spreadsheet program and use the first two rows from the worksheet to record the frequency mass function. See figures 8-10.

The Poisson random variable has an infinite support. One way to work with such a class of distributions is to truncate the support whenever the cumulative distribution function approaches the unit within a certain tolerance level. Figure 9 illustrates this idea to the tolerance level customized in our spreadsheet package.

We now use the first column to keep track of the values in the support of the random variable of aggregate claims. Each unit represents one billion dollars[6]. To know where

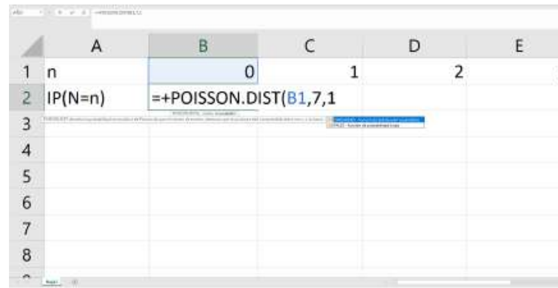

Figure 8 Actually, the prompt `POISSON.DIST(B1,7,1)` *does not* yield the Poisson mass function at $B1 = 0$ with mean $\lambda = 7$, but the cumulative distribution function at $B1 = 0$ with mean $\lambda = 7$. The mass function at $B1 = 0$ with mean $\lambda = 7$ is given by the syntax `POISSON.DIST(B1,7,0)`. We will revise this situation later.

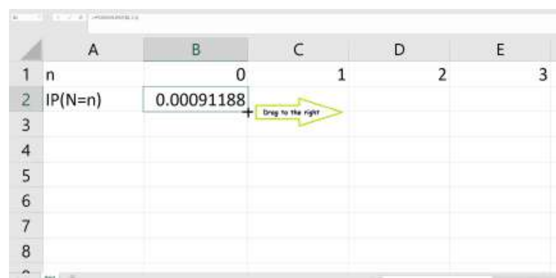

Figure 9 Drag the formula from cell `B2` until you get a result of 1. Depending on the customized precision of your spreadsheet program, this will happen at $x = 21$ or $x = 27$.

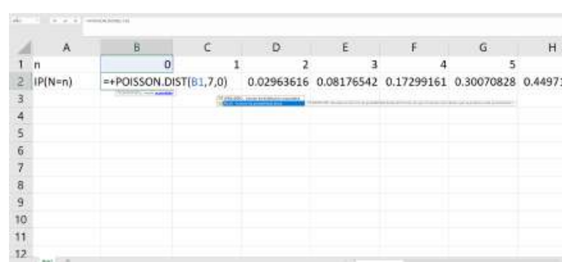

Figure 10 Change the final argument in the prompt from figure 8 so that the formula `POISSON.DIST(B1,7,0)` is displayed in all the cells from the second row.

[5] According to Stoltzfus and Dalton (2010), in a Gamma-like model, the standard deviation is proportional to the mean. See page II-67 in their book.

[6] Note that this stepsize is *gargantuan*. It is not common at all to use numbers of this sort as stepsizes. However, we do it here to dramatize the illustration. In a real-life application, it is more common to use smaller stepsizes. However, the main point here is the fact that the sums (resp. integrals) of the mass (resp. cumulative distribution) functions add-up to one.

to truncate the (infinite) support of the random variable of aggregate claims, we turn to Chebyshiov's inequality (see Chapter II.4.4 in the book by Mood *et al.* (1974)):

$$\mathbb{P}\left(|S - \mathbb{E}S| \le k\sqrt{\mathbf{var}S}\right) \le \frac{1}{k^2} \text{ for } k > 0.$$

Then we know the truncation point s^* must be at least equal to

$$\begin{aligned} \mathbb{E}S + k\sqrt{\mathbf{var}S} &= 756 + k\sqrt{86,184} \\ &\approx 756 + 293.5711157k \qquad (10) \end{aligned}$$

to bound the error in our estimation of $\mathbb{P}(S \le s^*)$ by $1/k^2$. Since we will not tolerate an error larger than $1/100$, we set $k = 10$ in (10) and obtain $s^* > 3691.711157$. See figure 11.

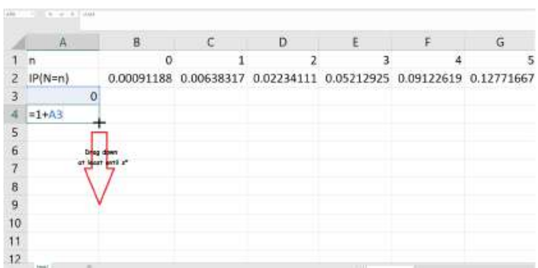

Figure 11 We use the first column to record the values in the support of the random variable of aggregate claims S.

We will use the remainder of the columns to keep track of the *conditional probability* of the random variable $[S|N = n]$, where n appears at the first row of the spreadsheet. With this in mind, column `B` will display conditional probabilities of the random variable $[S|N = 0]$, column `C` will show conditional probabilities of the random variable $[S|N = 1]$, column `D` will show conditional probabilities of the random variable $[S|N = 2]$, and so on, until column `AC`, where we will place the conditional probability of the random variable $[S|N = 27]$.

We start by recording the *mass* function of the random variable $[S|N = 0]$. Note that, in the present context, it is impossible for S to be greater than zero if $N = 0$. Then, we type a one, and many zeroes in column `B`. See figure 12 (the symbol $\chi_A(s)$ stands for an indicator function of the set A evaluated at s).

To compute the conditional *density* function of the random variable $[S|N = 1]$, we note that, by assumption **H1**, this is just the density function of the random variable of

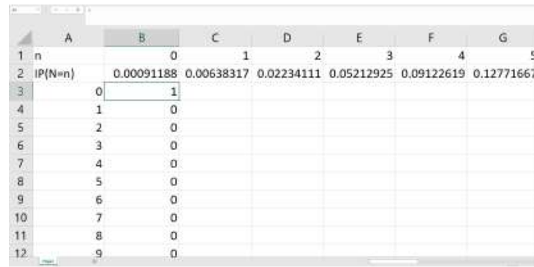

Figure 12 The conditional mass function of the random variable $[S|N = 0]$ is given by $\mathbb{P}(S = s|N = 0) = \chi_0(s)$.

aggregate claims S. However, we note that, by theorem A.2, it is also a Gamma random variable with shape parameter equal to $n \times \alpha = 1 \times 18$, and shape parameter of 6. Thus, we input the prompt

```
=DISTR.GAMMA.N($A3,C$1*18,6,0)
```

into cell `C3` and drag the result all the way down (at least up to $s^* = 3692$). See figure 13.

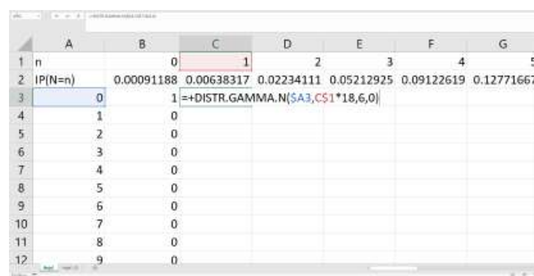

Figure 13 Observe we are no longer recording a *mass* function, but a density function. Moreover, we are fixing the column in the first argument of the prompt; and the row, in the second. Our purpose is to drag this formula down, but also to the right.

To generate the rest of the conditional density functions, we select the range with the conditional density function of the random variable $[S|N = 1]$ (*i.e.* in our case, range `C3:C3742`[7]), copy it, and paste it in the range `D3:AC3`. See figure 14.

Yet another application of the theorem of total probability gives us the density function of the random variable of aggregate claims, along with the point of mass

[7] Note that we selected an endpoint of the support of the random variable of $3739 > 3691.71157 = s^*$. Indeed, the 3742nd row corresponds to a value of 3739 in the support of S.

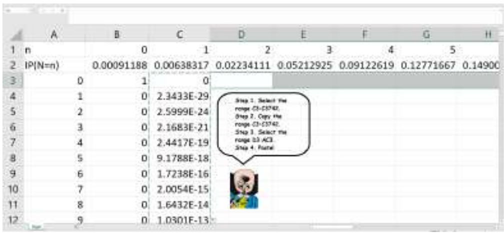

Figure 14 Uatu the Watcher (as depicted by Byrne (2005)) tells us how to compute the conditional density functions for the random variables $[S|N = n]$, with $n = 2, ..., 27$.

at $s = 0$. (For example, (7) gives us the cumulative distribution function of the random variable of aggregate claims evaluated at s.) Figure 15 shows how to obtain $f_S(1) = \sum_{n=0}^{\infty} f_{[S|N=n]}(1)\mathbb{P}(N = n)$. This number represents the *density* of the random variable S evaluated at 1. The entry in cell AD3 represents the *probability* that $\mathbb{P}(S = 0)$.

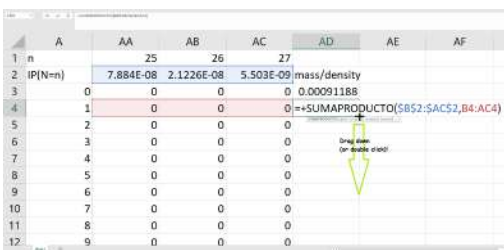

Figure 15 The blue range represents the mass function of the random variable for the frequency N, while the red range represents the mass function of the conditional random variable $[S|N = 0]$, and the density function of the conditional random variable $[S|N = n]$ for $n = 1, ...$ Note, however, that only the entry in the cell AD3 represents a probability, the remainder of the entries in column AD stand for densities.

Column AD represents the mass/density of the mixed-type random variable of aggregate claims S. Figure 16 is its geometric depiction.

As a means of verification, now we use the entries of range AD3:AD3742 to approximate the mean of the random variable of aggregate claims S from (8). To do this, we compute the product of the values in the support

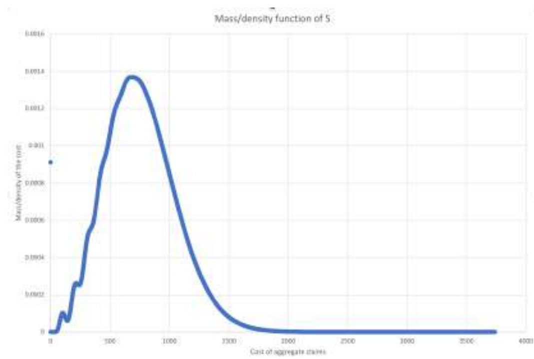

Figure 16 Note that the point of mass at $s = 0$ is isolated from the rest of the points.

of S, with the mass/densities from column AD3:AD3742[8]. We use the cell AD1 to this end. See figure 17.

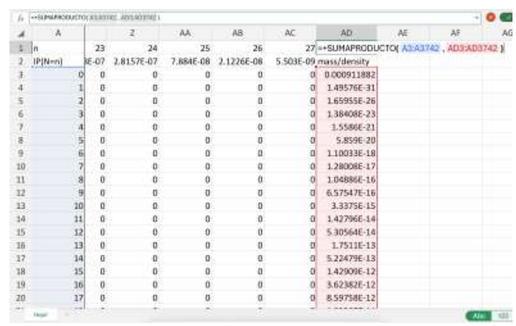

Figure 17 Our approximation should be very close to $\mathbb{E}S = 756$.

Now, we approximate the expected losses of each agent. To do Iron Man's, we use column AE to keep track of his losses as a function of the support of the random variable S. Since his *limit of responsibility* is 350 billion dollars, we use the original support of the random variable of aggregate claims and input the prompt

```
=IF(A3<=350,A3,350)
```

[8] This is an approximation of the expectation from (8): we are doing

$$0 \times \mathbb{P}(S = 0) + \sum_{s=1}^{\infty} s f_S(s),$$

when we should be doing

$$\mathbb{E}S = 0 \times \mathbb{P}(S = 0) + \int_0^{\infty} s f_S(s) \mathrm{d}s.$$

The reason for this is that the sum in the former expression is a Riemann approximation (with unitary stepsize) to the integral in the latter.

in cell AE3 and press “Return”. Then we double-click in the right-hand-side corner of the cell AE3. See figure 18.

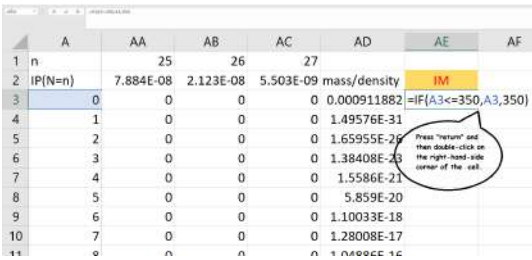

Figure 18 We place the support of the random variable of Iron Man’s expenses in column AE.

Next, proceed as in the computation shown in figure 17. This will generate the expected loss of Iron Man[9]. See figure 19.

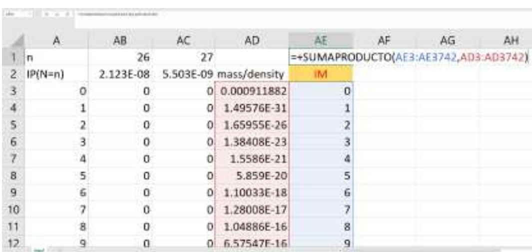

Figure 19 The result should be close to 343.2994774. Do you think Iron Man will choose to pay this much instead of risking paying up to 350 billion USD?

To calculate Batman’s expected loss, we start by considering the support of the share of the risk he is taking on. To this end, use column AF and record the support by prompting

```
=IF(A3<=350,0,IF(A3<=1000,A3-350,650))
```

in cell AF3 and press “Return”. Then we double-click in the right-hand-side corner of the cell AF3. See figure 20.

We compute Batman’s expected loss by approximating the corresponding Riemann sum using the prompt displayed in figure 21.

We complete our analysis by computing the expected loss of the governments of the world. We use an analogous approach to the ones we used in figures 18-19

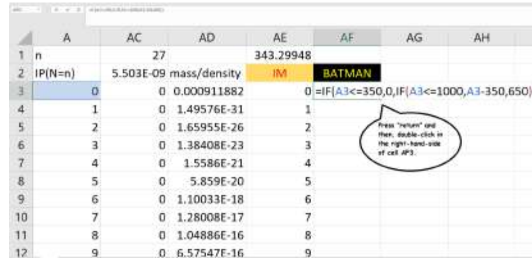

Figure 20 We place the support of the random variable of Batman’s expenses in column AF. One way to see why we typed in that particular prompt is: what is the payment issued by the Dark Knight when the overall expenses are below 350 billion?, and what if the overall expenses are between 350 billion and one trillion? What happens after one trillion dollars?

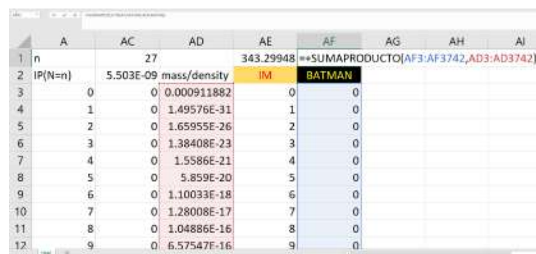

Figure 21 The result should be close to 374.8962607. Do you think Batman will choose to pay this much instead of risking covering the expenses that exceed 350 billion, and up to one trillion?

and 20-21. Figure 22 displays our results. Do you think the governments of the world would agree to pay 37.80425635 billion instead of risking covering the expenses that surpass one trillion dollars?

Is it expensive?

The last subsection concluded with three questions:

- Do you think Iron Man will agree to pay 343.2994774 instead of risking paying up to 350 billion USD?
- Do you think Batman will agree to pay 374.8962607 billion instead of risking covering the expenses that exceed 350 billion, and up to one trillion USD?
- Do you think the governments of the world will agree to pay 37.80425635 billion instead of risking covering the expenses that surpass one trillion dollars?

[9] Recall that this approach works only because we are approximating a Riemann sum.

	A	AC	AD	AE	AF	AG	AH
1	n	27		343.29948	374.89626	37.804256	
2	IP(N=n)	5.503E-09	mass/density	IM	BATMAN	[illegible]	
3	0	0	0.000911882	0	0	=IF(A3>1000,A3-1000,0)	
4	1	0	1.49576E-31	1	0	0	
5	2	0	1.65955E-26	2	0	0	
6	3	0	1.38408E-23	3	0	0	
7	4	0	1.5586E-21	4	0	0	
8	5	0	5.859E-20	5	0	0	
9	6	0	1.10033E-18	6	0	0	
10	7	0	1.28008E-17	7	0	0	
11	8	0	1.04886E-16	8	0	0	
12	9	0	6.57547E-16	9	0	0	

Figure 22 This is the summary of two steps: the computations of the support of the risk share taken on by the governments of the world, and the Riemann sum corresponding to the expected loss of such a random variable. Note the use of the prompts `=IF(A3>1000,A3-1000,0)` and `=SUMPRODUCT(AG3:AG3742,AD3:AD3742)` in cells `AG3` (and below), and `AG1`, respectively. Can you guess what is the sum of the numbers from the entries in range `AE1:AG1`? (Hint: recall figure 17.)

Attemping to answer to these questions without any more information is a futile exercise. Indeed, except for the first number (which seems rather large in comparison to what it is supposed to cover), these expected losses do not provide a context to assess whether they are expensive or not.

To try to give proper answers to the questions, we compute the probabilities that the agents end up spending more than the expected losses should they avoid paying these amounts. For Iron Man, calculate the *sum*[10] of the probabilities from 344 billion USD and on in the support of the original random variable of aggregate claims. That is, the sum of the entries in the range `AD347:AD3742`. The result is 93.17%. This means that the probability that Tony Stark spends less than a risk premium of 343.2994774 billion is of barely 6.83%. Will he agree then to pay *only* this risk premium? Our guess is: no! In his own words, he is a genius, billionaire, playboy, philanthropist. Besides, he is *the* Iron Man... if there is a 6.83%-chance of losing less than 343.2994774 billion, why not take it?

As for Mr. Wayne, we proceed analogously and compute the probability that he ends up losing more than a risk premium of 374.8962607 billion. To this end, we add up[11] all the probabilities from 350+374.8962607 billion USD to obtain that the probability that Bruce Wayne spends more than 374.8962607 billion USD equals 51.55%. That is, the sum of the entries in the range `AD728:AD3742` is 51.55%. (To see this, note that the first 350 billion would be covered by Tony Stark, and it should be Batman the one who pays the claims above this amount, and up to one trillion dollars.) Will the *greatest world detective* agree to pay this much? We think he would! After all, we are talking about the one superhero who devised contingency plans in case a member of the *Justice League of America* went rogue (an event that occurred only in the events narrated by Kirby *et al.* (2005); Waid *et al.* (2001); Snyder *et al.* (2014) and Zdarsky and Jiménez (2022) –of course, in the main continuity comics- during the last 85 years). What are the odds of that?!

The governmets of the world would definitely pay the risk premium of 37.8 billion USD. That is, although the probability that they end up spending more than this much is of only 16.74% (the sum of the probabilities corresponding to 1000+37.8 billion USD in the support of the original random variable of aggregate claims), a government would not risk a catastrophe with a margin larger than 1%.

5. CONCLUDING REMARKS

This paper gives us a glimpse on how to use two tools from the theory of risks to prevent a *civil war* among the superheroes. Namely, the expectation of the random variable of aggregate claims and the cumulative distribution function. This is, of coruse, an overstatement for two reasons: there are no superheroes in real life, and the actuarial techniques involved in the pricing of an insurance plan only start where we conclude our manuscript. Moreover, we have oversimplified many of the assumptions and methods used in a real problem. For example, our stepsizes are huge, there is no statistical verification of the distribution of the involved random variables, and we substitute Riemann integrals for simple sums. However, we believe this paper represents a good practice for students of actuarial sciences, for we profit from a context familiar to many of our pupils to teach them the basics of some important topics in the risk and probability theories.

To complete the presentation in the same fashion as the used to started it, we will leave the reader with a final question. In the example developed here, the random variable we used to model the frequency followed the Poisson law. What if the Poisson parameter from the frequency random variable is *radiated* with a Gamma distribution? See figure 23, and appendices A and B.

[10] Or more properly, the *Riemann integral*.
[11] Or more adequately, "we compute the Riemann integral of".

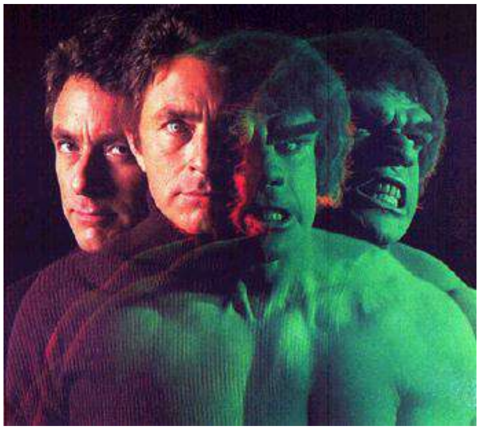

Figure 23 What if the Poisson parameter follows a Gamma distribution? Image source: The TV series produced by Johnson (1978).

A. THE GAMMA RANDOM VARIABLE

Define the *Gamma function* as: $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x$ for $\alpha > 0$. The integration-by-parts formula to yields

$$\Gamma(\alpha) = (\alpha-1)\cdot\Gamma(\alpha-1) \tag{11}$$

for $\alpha > 1$, and $\Gamma(1) = 1$. Indeed, let $u = x^{\alpha-1}$ and $\mathrm{d}v = \mathrm{e}^{-x}\mathrm{d}x$. Computing, respectively the derivative and anti-derivative of these expressions, and substituting them into $\int_0^\infty u\mathrm{d}v = uv|_0^\infty - \int_0^\infty v\mathrm{d}u$ yields

$$\Gamma(\alpha) = -x^{\alpha-1}\mathrm{e}^{-x}\Big|_0^\infty + (\alpha-1)\int_0^\infty x^{\alpha-2}\mathrm{e}^{-x}\mathrm{d}x.$$

Since the last integral equals $\Gamma(\alpha-1)$, this gives (11). Note that, when α belongs to the set of integer numbers, (11) becomes the recursive definition of factorial.

Now let $\alpha > 0$ and $\theta > 0$. A change of variable leaves us with:

$$\int_0^\infty x^{\alpha-1}\mathrm{e}^{-x/\theta}\mathrm{d}x = \theta^\alpha\Gamma(\alpha). \tag{12}$$

Proof. To see this, let $y := x/\theta$. Then $\mathrm{d}y = \frac{1}{\theta}\mathrm{d}x$. The substitution of these expressions into the left-hand side of (12) yields

$$\int_0^\infty (\theta y)^{\alpha-1}\,\mathrm{e}^{-y}\theta\mathrm{d}y = \theta^\alpha\int_0^\infty x^{\alpha-1}\mathrm{e}^{-y}\mathrm{d}y = \theta^\alpha\Gamma(\alpha).$$

The last equality holds by (11). □

Define the *Gamma density function*[12]

$$f_X(x) := \frac{1}{\theta^\alpha\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-x/\theta}. \tag{13}$$

for $x > 0$, with $\alpha > 0$ and $\theta > 0$. Note that f_X is a density function:

- The function $f_X(x)$ is positive for all $x > 0$.

  Proof. To see the result, it suffices to see that the two rightmost factors in the definition (13) are both positive for all $x > 0$, while $\theta^\alpha\Gamma(\alpha) > 0$. □

- The integral $\int_0^\infty f_X(x)\mathrm{d}x$ equals the unit.

  Proof. This is an immediate consequence of (12). Indeed, it suffices to note that (13) consists of the integrand in the left-hand side of (12), and the expression in the right-hand side of (12). □

As an illustration, we provide table 1 with the first ten values of $f_X(x)$ for $x > 0$ with $\alpha = 10.5$, $\theta = 2.5$ and a stepsize of 0.1.

x	$f_X(x)$
0.1	7.5044×10^{-19}
0.2	5.2207×10^{-16}
0.3	2.3617×10^{-14}
0.4	3.4896×10^{-13}
0.5	2.7928×10^{-12}
0.6	1.5167×10^{-11}
0.7	6.3026×10^{-10}
0.9	6.3336×10^{-10}
1	1.6557×10^{-09}

Table 1 The first ten values of x and $f_X(x)$ for $x > 0$ with $\alpha = 10.5$, $\theta = 2.5$ and a stepsize of 0.1.

Figure 24 is a representation of the first ±1000 ordered pairs obtained above with $\alpha = 10.5$ and $\theta = 2.5$.

We recommend that the reader fix the *scale parameter* θ and change the *shape parameter* α. What if one does it the other way around? Precisely in this context, if we fix θ

[12] Note that there is a difference between the Gamma *function*, and the Gamma *density function*.

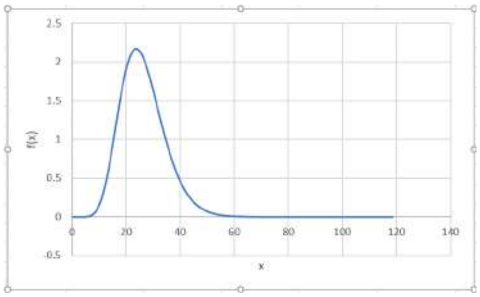

Figure 24 Dispersion plot of x *vs.* $f_X(x)$.

at any given level and let $\alpha \to \infty$. The resulting density would look a lot like the Gaussian bell (see, for instance the chapter written by López-Barrientos *et al.* (2022)).

Let $c > 0$ be a given constant and $f_X(x)$ be the Gamma density function defined in (13) with known shape parameter $\alpha > 0$, and scale parameter $\theta > 0$. The theorem of change of variable (see section V.5 in the book by Mood *et al.* (1974)) can help us find the density function of the random variable $Y := cX$.

Theorem A.1. *Let X be a Gamma random variable with shape parameter $\alpha > 0$ and scale θ. The random variable $Y := cX$ has a Gamma distribution with shape parameter $\alpha > 0$ and scale parameter $c\theta$.*

Proof. For $y > 0$,

$$\begin{aligned} f_Y(y) &= \frac{1}{\theta^\alpha \Gamma(\alpha)} \left(\frac{y}{c}\right)^{\alpha-1} \mathrm{e}^{-y/(c\theta)} \cdot \left| \frac{\mathrm{d}}{\mathrm{d}y} \frac{y}{c} \right| \\ &= \frac{1}{(c\theta)^\alpha \Gamma(\alpha)} y^{\alpha-1} \mathrm{e}^{-y/(c\theta)}. \end{aligned}$$

This is the density function of a Gamma random variable with shape parameter α and scale parameter $c\theta$. □

Let $f_X(x)$ be the Gamma density function defined in (13) with known shape parameter $\alpha > 0$, and scale parameter $\theta > 0$. The moment generating function of the Gamma random variable X is given by

$$M_X(t) = (1 - \theta t)^{-\alpha}. \tag{14}$$

Proof. We make

$$\begin{aligned} M_X(t) &= \mathbb{E}\left[\mathrm{e}^{tX}\right] = \int_0^\infty \mathrm{e}^{tx} f_X(x) \mathrm{d}x \\ &= \int_0^\infty \mathrm{e}^{tx} \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-x/\theta} \mathrm{d}x \\ &= \int_0^\infty \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-x\left(\frac{1}{\theta} - t\right)} \mathrm{d}x. \end{aligned} \tag{15}$$

Analogously to the proof of (12), we let $y := x\left(\frac{1}{\theta} - t\right)$, so that $\mathrm{d}y = \left(\frac{1}{\theta} - t\right)\mathrm{d}x$. Substituting these two expressions into (15) yields

$$\begin{aligned} M_X(t) &= \int_0^\infty \frac{1}{\theta^\alpha \Gamma(\alpha)} \left(\frac{y}{\frac{1}{\theta} - t}\right)^{\alpha-1} \mathrm{e}^{-y} \frac{1}{\frac{1}{\theta} - t} \mathrm{d}y \\ &= \frac{1}{\theta^\alpha \Gamma(\alpha)} \left(\frac{1}{\frac{1}{\theta} - t}\right)^\alpha \int_0^\infty y^{\alpha-1} \mathrm{e}^{-y} \mathrm{d}y \\ &= \frac{1}{\theta^\alpha \Gamma(\alpha)} \left(\frac{1}{\frac{1}{\theta} - t}\right)^\alpha \Gamma(\alpha) \\ &= \frac{1}{\theta^\alpha} \left(\frac{1}{\frac{1}{\theta} - t}\right)^\alpha \\ &= \frac{1}{\theta^\alpha} \left(\frac{\theta}{1 - \theta t}\right)^\alpha \\ &= (1 - \theta t)^{-\alpha}. \end{aligned}$$

This completes the proof. □

From (14), it is not hard to see that

$$\begin{aligned} \mathbb{E}X &= \alpha\theta, & (16) \\ \mathbf{var}X &= \alpha\theta^2. & (17) \end{aligned}$$

Theorem A.2. *If X_i is Gamma-distributed with parameters of shape $\alpha_i > 0$, and of scale $\theta > 0$ for $i = 1, ..., n$, then*

$$X_1 + \cdots + X_n$$

is Gamma-distributed with parameters of shape $\alpha_1 + \cdots + \alpha_n$, and scale $\theta > 0$.

Proof. Since, for $i = 1, ..., n$, X_i is Gamma-distributed with parameters of shape $\alpha_i > 0$, and scale $\theta > 0$ then $M_{X_i}(t) = (1 - \theta t)^{-\alpha_i}$. Using the moment generating function technique from chapter V.4 in the book by Mood *et al.* (1974), we obtain

$$\begin{aligned} M_{X_1 + \cdots + X_n}(t) &= (1 - \theta t)^{-\alpha_1} \cdots (1 - \theta t)^{-\alpha_n} \\ &= (1 - \theta t)^{-(\alpha_1 + \cdots + \alpha_n)}. \end{aligned}$$

This matches the moment generating function of a Gamma distribution with parameters of shape $\alpha_1 + \cdots + \alpha_n$, and scale $\theta > 0$. □

From theorem A.2, we know that the distribution of the sum of $c > 0$ independent and identically Gamma-distributed random variables with parameters of shape $\alpha > 0$, and scale $\theta > 0$ is Gamma with parameters of shape $c\alpha$, and scale $\theta > 0$. Note that this does not contradict theorem A.1, because $cX \neq X_1 + \cdots + X_c$, even if $X_i \sim \Gamma(\alpha, \theta)$ for $i = 1, ..., c$.

B. THE NEGATIVE BINOMIAL RANDOM VARIABLE

This bonus section illustrates the derivation of the Negative Binomial random variable from the Poisson random variable. It presents an example used in the author's class on Risk Theory. Appendix B.2 in the book by Klugman *et al.* (2019) presents a different (but swift) approach to this and other random variables of the so-called class $(a, b, 0)$.

A swan lays a Poisson number of eggs with parameter Λ, where Λ is a Gamma random variable with mean α/β and variance α/β^2. A cygnet hatches out of each egg with probability p, regardless of the other eggs.

The mean of the number of eggs laid by the swan is $\frac{\alpha}{\beta}$.

Proof. Let N be the number of eggs laid by the swan. An application of the theorem of the nested expectation results in $\mathbb{E}N = \mathbb{E}[\mathbb{E}(N|\Lambda)] = \mathbb{E}\Lambda = \frac{\alpha}{\beta}$. □

The variance of the number of eggs laid by the swan is $\frac{\alpha(\beta+1)}{\beta^2}$.

Proof. An invocation of the theorem of nested variance yields (see, for instance, formula (2.2.11) in the book by Bowers *et al.* (1997)):

$$\begin{aligned}
\mathbf{var}N &= \mathbb{E}[\mathbf{var}(N|\Lambda)] + \mathbf{var}[\mathbb{E}(N|\Lambda)] \\
&= \mathbb{E}\Lambda + \mathbf{var}\Lambda \\
&= \frac{\alpha}{\beta} + \frac{\alpha}{\beta^2} \\
&= \frac{\alpha(\beta+1)}{\beta^2}.
\end{aligned}$$

□

The mass function of the number of eggs is Negative Binomial with parameters α and $\frac{1}{\beta+1}$.

Proof. Considering that Gamma density function with parameters of shape α and scale $\frac{1}{\beta}$ is given by

$$f_\Lambda(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\lambda\beta} \lambda^{\alpha-1}.$$

The theorem of total probability yields

$$\begin{aligned}
\mathbb{P}(N = n) &= \int_0^\infty \mathbb{P}(N = n|\Lambda = \lambda) f_\Lambda(\lambda) d\lambda \\
&= \int_0^\infty \frac{\lambda^n}{n!} e^{-\lambda} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\lambda\beta} \lambda^{\alpha-1} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha) n!} \int_0^\infty \lambda^{n+\alpha-1} e^{-\lambda(1+\beta)} d\lambda.
\end{aligned}$$

The change of variable $z := \lambda(1+\beta)$ (with $dz = (1+\beta)d\lambda$) yields:

$$\begin{aligned}
\mathbb{P}(N = n) &= \frac{\beta^\alpha}{n!\Gamma(\alpha)} \int_0^\infty \left(\frac{z}{1+\beta}\right)^{n+\alpha-1} e^{-z} \frac{dz}{1+\beta} \\
&= \frac{\beta^\alpha}{n!\Gamma(\alpha)} \left(\frac{1}{1+\beta}\right)^{n+\alpha} \int_0^\infty z^{n+\alpha-1} e^{-z} dz \\
&= \frac{\beta^\alpha}{n!\Gamma(\alpha)} \left(\frac{1}{1+\beta}\right)^{n+\alpha} \Gamma(n+\alpha) \\
&= \frac{(\alpha+n-1)\cdots\alpha}{n!} \left(\frac{\beta}{\beta+1}\right)^\alpha \left(\frac{1}{\beta+1}\right)^n
\end{aligned}$$

for $n = 0, 1, ...$ □

The expected number of cygnets that hatch out of the eggs, given that the swan laid N eggs is pN.

Proof. Let K be the number of cygnets that hatch out the eggs. We need to compute $\mathbb{E}(K|N)$. Since $[K|N]$ is a Binomial random variable with parameters N and p, then $\mathbb{E}[K|N] = pN$. □

The expected number of cygnets that hatch out of the eggs is $\frac{\alpha}{\beta}$.

Proof. Let K be the number of cygnets that hatch out the eggs. A new application of the theorem of the nested expectation yields $\mathbb{E}[\mathbb{E}(K|N)] = p\mathbb{E}N = p\frac{\alpha}{\beta}$. □

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